SMALL PERTURBATIONS OF A LAMINAR BOUNDARY LAYER

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UDC 532.526.3

In [1] a general classification scheme was presented for spatially local laminar flows, and the major studies of spatially local flows were noted.

The present study will consider the compensation flow regime [2], in which the infrasonic portion of a wall boundary layer near a body is perturbed. Various cases of flow in perturbed regions near thin spatial roughness on a body surface are distinguished, and numerical solutions are obtained in the linear approximation. It is demonstrated that the form of propagation of a pressure perturbation depends on the form of the roughness and the ratio of its characteristic width and length.

1. We will consider passage of a uniform flow of viscous gas at Mach numbers $M_\infty^2 - 1 \sim O(1)$ over a plane plate, upon the surface of which at a distance ℓ from the leading edge there is located a small spatial roughness, with the Reynolds number being large, but subcritical, $Re_\infty = \rho_\infty u_\infty \ell/\mu_\infty = \varepsilon^{-2}$. We will construct a steady-state solution of the Navier–Stokes equation as $\varepsilon \to 0$. In the future we will use only dimensionless variables, referring all linear dimensions to ℓ, the pressure to $\rho_\infty u_\infty^2$, and the enthalpy to u_∞^2, with remaining flow functions being referenced to their values in the uniform incident flow.

Concerning the characteristic dimensions of the small roughness, we assume that its thickness a is less than the characteristic boundary layer thickness on the plate at this point $[a < \delta \sim O(\varepsilon)]$, while its length b ($a < b \lesssim 1$) and width c ($a < c$), i.e., we study flow around a small rough spot, located on the "bottom" of the boundary layer on the plate.

We will consider the compensation regime of flow over small spatial roughness [1, 3, 4], where essentially only the infrasonic shear portion of the boundary layer in close proximity to the plate is perturbed. We will analyze the most general case, in which the perturbed region 3 has characteristic dimensions $x \sim O(b)$, $y \sim O(a)$, and $z \sim O(c)$, i.e., equal in order of magnitude to the characteristic dimensions of the roughness itself, viscosity is significant, and nonlinear perturbations of the longitudinal velocity $u - u \sim \Delta u$ are introduced. Then, considering that in the boundary layer the estimate $u \sim O(y/\varepsilon)$ is valid, we have

$$a \sim O(\varepsilon b^{1/3}),\ \varepsilon^{3/2} < a < b,\ c,\ u \sim \Delta u \sim O(b^{1/3}). \tag{1.1}$$

The small roughness then introduces perturbations of shear stress and thermal flux equal in order of magnitude to the values within the boundary layer at the plate surface themselves.

2. For roughness which is not too narrow ($b \lesssim c$) in region 3 from the equations of conservation of longitudinal and transverse momentum and continuity, using Eq. (1.1) we can obtain estimates for the perturbations in pressure, transverse, and vertical velocities:

$$\Delta p \sim O(b^{2/3}),\ w \sim O(b^{4/3}/c),\ v \sim O(\varepsilon/b^{1/3}). \tag{2.1}$$

If the pressure perturbation is created by interaction of the roughness with a uniform incident flow, then $\Delta p \sim O(a/b)$ and comparing this estimate to Eq. (2.1), it is simple to see that the compensation regime of flow over nonnarrow roughness is realized for

$$a \sim O(\varepsilon b^{1/3}),\ \varepsilon^{3/2} < b < \varepsilon^{3/4},\ b \leqslant c. \tag{2.2}$$

Equations (1.1), (2.1), and (2.2) allow us to introduce in region 3 the following independent variables and asymptotic expansions of the flow functions:

$$\begin{gathered} x = bx_3,\ y = \varepsilon b^{1/3} y_3,\ z = cz_3, \\ u = b^{1/3}u_3 + \ldots,\ v = (\varepsilon/b^{1/3})v_3 + \ldots,\ w = (b^{4/3}/c)w_3 + \ldots, \\ \Delta p = b^{2/3}p_3 + \ldots,\ h = h_w + b^{1/3}h_3 + \ldots,\ \rho = \rho_w + \ldots,\ \mu = \mu_w + \ldots \end{gathered} \tag{2.3}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 69-79, September-October, 1987. Original article submitted April 28, 1986.

 0021-8944/87/2805-0706$12.50

with the subscript w referring to quantities in the boundary layer at the plate surface at the point where the roughness is located.

Substituting the expansions of Eq. (2.3) into the Navier–Stokes equations and performing the limiting transition as $\varepsilon \to 0$, $\varepsilon^{3/2} < b \leq c$ shows that in the first approximation the flow in region 3 is described by the equations of a spatial Prandtl boundary layer for an incompressible gas

$$\frac{\partial u_3}{\partial x_3}+\frac{\partial v_3}{\partial y_3}+\left(\frac{b}{c}\right)^2\frac{\partial w_3}{\partial z_3}=0,\quad \rho_w\left(u_3\frac{\partial u_3}{\partial x_3}+v_3\frac{\partial u_3}{\partial y_3}+\left(\frac{b}{c}\right)^2 w_3\frac{\partial u_3}{\partial z_3}\right)+\frac{\partial p_3}{\partial x_3}=\mu_w\frac{\partial^2 u_3}{\partial y_3^2}, \tag{2.4}$$

$$\rho_w\left(u_3\frac{\partial w_3}{\partial x_3}+v_3\frac{\partial w_3}{\partial y_3}+\left(\frac{b}{c}\right)^2 w_3\frac{\partial w_3}{\partial z_3}\right)+\frac{\partial p_3}{\partial z_3}=\mu_w\frac{\partial^2 w_3}{\partial y_3^2},\quad \frac{\partial p_3}{\partial y_3}=0,$$

$$\rho_w\left(u_3\frac{\partial h_3}{\partial x_3}+v_3\frac{\partial h_3}{\partial y_3}+\left(\frac{b}{c}\right)^2 w_3\frac{\partial h_3}{\partial z_3}\right)=\frac{\mu_w}{\Pr}\frac{\partial^2 h_3}{\partial y_3^2}$$

(where Pr is the Prandtl number). On the surface of the roughness $y_3 = f(x_3, z_3)$ the usual adhesion and nonpenetration conditions must be satisfied:

$$u_3 = v_3 = w_3 = h_3 = 0 \quad (y_3 = f(x_3, z_3)). \tag{2.5}$$

Initial boundary conditions as $x_3 \to -\infty$ or $z_3 \to \pm\infty$ can be found by merging the solution from region 3 with the near-the-wall portion of the boundary layer on the plate:

$$u_3 \to Ay_3,\ h_3 \to By_3,\ v_3,\ w_3,\ p_3 \to 0 \quad (x_3 \to -\infty); \tag{2.6}$$

$$u_3 \to Ay_3,\ h_3 \to By_3,\ v_3,\ w_3,\ p_3 \to 0 \quad (z_3 \to \pm\infty). \tag{2.7}$$

Here $A = (\partial u_0/\partial y_2)_w$; $B = (\partial h_0/\partial y_2)_w$; $y_2 = y/\varepsilon$, $u_0(y_2)$, and $h_0(y_2)$ are the longitudinal velocity and enthalpy profiles in the boundary layer on the plate.

To find the external boundary conditions as $y_3 \to \infty$, it is necessary to consider perturbed region 2, the characteristic thickness of which $y \sim O(b)$ at $\varepsilon^{3/2} < b < \varepsilon$ or $y \sim O(\varepsilon)$ at $\varepsilon \leq b < \varepsilon^{3/4}$, i.e., is "fatter" than region 3. Therefore in region 2 in the first case we introduce independent variables and asymptotic expansions of the flow functions

$$\begin{gathered} x_2 = x_3 = x/b,\ y_2 = y/b,\ z_2 = z_3 = z/c,\\ u = (b/\varepsilon)Ay_2 + (\varepsilon/b^{1/3})u_2 + \ldots,\ v = (\varepsilon/b^{1/3})v_2 + \ldots,\\ w = (\varepsilon b^{2/3}/c)w_2 + \ldots,\ \Delta p = b^{2/3}p_2 + \ldots,\ \rho = \rho_w + \ldots,\\ h = h_w + (b/\varepsilon)By_2 + (\varepsilon/b^{1/3})h_2 + \ldots, \end{gathered} \tag{2.8}$$

while in the second case we have variables and an expansion of the form

$$\begin{gathered} x_2 = x_3 = x/b,\ y_2 = y/\varepsilon,\ z_2 = z_3 = z/c,\\ u = u_0(y_2) + b^{2/3}u_2 + \ldots,\ v = (\varepsilon/b^{1/3})v_2 + \ldots,\\ w = (b^{5/3}/c)w_2 + \ldots,\ \Delta p = b^{2/3}p_2 + \ldots,\\ \rho = \rho_0(y_2) + b^{2/3}\rho_2 + \ldots,\ h = h_0(y_2) + b^{2/3}h_2 + \ldots \end{gathered} \tag{2.9}$$

[$\rho_0(y_2)$ is the density profile in the boundary layer on the plate]. Substitution of the expansions of Eq. (2.8) or (2.9) in the Navier–Stokes equation and performance of the limiting transition as $\varepsilon \to 0$, $\varepsilon^{3/2} < b < \varepsilon^{3/4}$, $b \leq c$ shows that in both cases the flow in region 2 will be described in the first approximation by Euler equations linearized relative to the incident flow [$u = (b/\varepsilon)Ay_2$ or $u = u_0(y_2)$], whence

$$A\rho_w v_2 + \partial p_2/\partial x_2 \to 0 \quad (y_2 \to 0). \tag{2.10}$$

Merging of the solutions in regions 2 and 3 with use of Eq. (2.10) gives the following external boundary conditions:

$$u_3 \to Ay_3,\ h_3 \to By_3,\ A\rho_w v_3 + \partial p_3/\partial x_3 \to 0,\ w_3 \to 0\ (y_3 \to \infty). \tag{2.11}$$

For the future, it will be convenient to take $b \sim c$ in Eqs. (2.4)-(2.7) and (2.11), describing the compensation flow regime over thin nonnarrow roughness (2.2), and to introduce new variables (without indices)

$$x_3 = b_1x,\ y_3 = a_1y,\ z_3 = c_1z, \tag{2.12}$$
$$u_3 = Aa_1u,\ v_3 = (Aa_1^2/b_1)\,v,\ w_3 = (Aa_1c_1/b_1)\,w,\ p_3 = \rho_w A^2a_1^2p,\ h_3 = Ba_1h,$$

where a_1, b_1, c_1 are the thickness, length, and width of the roughness in scales of a, b, c, respectively [for example, the physical dimensioned thickness of the roughness is equal to ℓaa_1, $a \sim O(\varepsilon b^{1/3})$, $a_1 \sim O(1)$]. In the variables of Eq. (2.12) system (2.4) takes on the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + C\frac{\partial p}{\partial x} = \Pi\frac{\partial^2 u}{\partial y^2}, \tag{2.13}$$
$$\frac{\partial p}{\partial y} = 0,\quad u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + D\frac{\partial p}{\partial z} = \Pi\frac{\partial^2 w}{\partial y^2},$$
$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z} = \frac{\Pi}{\Pr}\frac{\partial^2 h}{\partial y^2},\quad \Pi = \frac{\mu_w b_1}{\rho_w Aa_1^3};$$

with internal boundary conditions from Eq. (2.5)

$$u = v = w = h = 0\quad (y = f(x,\ Ez)); \tag{2.14}$$

while the external boundary conditions of Eq. (2.11) transform to

$$u,\ h \to y,\ w \to 0,\ v + C\partial p/\partial x \to 0\ (y \to \infty); \tag{2.15}$$

and initial conditions (2.6), (2.7) take on the form

$$u,\ h \to y,\ v,\ w,\ p \to 0\ (x \to -\infty); \tag{2.16}$$
$$u,\ h \to y,\ v,\ w,\ p \to 0\ (z \to \pm\infty). \tag{2.17}$$

Here the coefficients C, D, and E, expressing the ratios of the actual roughness measurements a_1, b_1, and c_1, take on the values

$$C = E = 1,\ D = (b_1/c_1)^2\ (b_1 \leqslant c_1) \tag{2.18}$$

while the form of the roughness f = (x, Ez) is normalized to unity with respect to height, width, and length.

It is obvious that the boundary problem of Eqs. (2.13)-(2.18) is applicable to study of the flow about various nonnarrow roughnesses ($b_1 \le c_1$). As $D \to 0$, $w \to 0$, and the solution will describe flow around wide roughnesses ($b_1 \ll c_1$) in planar sections z = const.

If in place of transformation (2.12) we use

$$x_3 = b_1x,\ y_3 = a_1\Pi^{1/3}y,\ z_3 = c_1z, \tag{2.19}$$
$$u_3 = Aa_1\Pi^{1/3}u,\quad v_3 = (Aa_1^2/b_1)\,\Pi^{2/3}v,\quad w_3 = (Aa_1c_1/b_1)\,\Pi^{1/3}w,$$
$$p_3 = \rho_w A^2a_1^2\Pi^{2/3}p,\quad h_3 = Ba_1\Pi^{1/3}h,$$

we can then write system (2.4) and internal boundary conditions (2.5) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + C\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \tag{2.20}$$
$$\frac{\partial p}{\partial y} = 0,\quad u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + D\frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial y^2},$$
$$u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z} = \frac{1}{\Pr}\frac{\partial^2 h}{\partial y^2};$$
$$u = v = w = h = 0\quad (y = \Pi^{-1/3}f(x,\ Ez)), \tag{2.21}$$

while conditions (2.6), (2.7), and (2.11) reduce to Eqs. (2.15)-(2.17).

For the variables used in boundary problems (2.13)-(2.18) or (2.15)-(2.18), (2.20), and (2.21) the dimensionless components of the shear stress τ_{xy}, τ_{yz} and thermal flux q can be expressed by

$$\tau_{xy} = \partial u/\partial y,\ \tau_{yz} = (c_1/b_1)\partial w/\partial y,\ q = \partial h/\partial y \tag{2.22}$$

while in the boundary layer on the plate near its surface $\tau_{xy} = q = 1$, $\tau_{yz} = 0$.

The similarity parameter Π characterizes the ratio of viscous layer thickness to thickness of the roughness. As $\Pi \to 0$ region 3 becomes nonviscous, and it is then convenient to use equations and boundary conditions (2.13)-(2.17). In this case it is necessary to examine in greater detail the viscous and thermally conductive sublayer near the surface of the roughness [5]. As $\Pi \to \infty$ the viscous layer is significantly thicker than the roughness and it is possible to linearize the boundary problem of Eqs. (2.15)-(2.17), (2.20), (2.21) with respect to the parameter $\lambda = \Pi^{-1/3} \ll 1$.

Differentiating the equation of conservation of longitudinal momentum, Eq. (2.20), with respect to x, and the equation for transverse momentum with respect to z, combining the results, and using boundary condition (2.21) for $p = p(x, z)$ we obtain a conventional Poisson equation

$$C\frac{\partial^2 p}{\partial x^2} + D\frac{\partial^2 p}{\partial z^2} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) \quad (y = \Pi^{-1/3} f(x, Ez)); \tag{2.23}$$

then differentiating the same equations with respect to y, we can eliminate p(x, z) from system (2.20)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,\quad \frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \frac{\partial^3 u}{\partial y^3}, \tag{2.24}$$
$$\frac{\partial}{\partial y}\left(u\frac{\partial w}{\partial y} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \frac{\partial^3 w}{\partial y^3},\quad u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z} = \frac{1}{\mathrm{Pr}}\frac{\partial^2 h}{\partial y^2}.$$

It is known [2] that the condition for determining pressure in the compensation regime of flow over roughness, Eq. (2.15),

$$v + C\partial p/\partial x \to 0 \quad (y \to \infty)$$

is equivalent to the condition

$$u = y + o(1) \quad (y \to \infty), \tag{2.25}$$

which indicates that for the flow regime studied change in thickness of the roughness is compensated by change in the thickness of the wall portion of the boundary layer on the plate. Therefore, the complete boundary problem of Eqs. (2.15)-(2.17), (2.20), (2.21) with the use of Eq.(2.25) in place of the condition for p(x, z) in Eq. (2.15) decomposes into two problems, the first of which with use of Eq.(2.24) permits determination of the velocity and enthalpy components, while the second gives the pressure distribution from Eq. (2.23). It should be noted that at values of the coefficients C, D, E given by Eq. (2.18) the first boundary problem does not depend on the roughness dimensions a_1, b_1, c_1, and its solution describes flow over a roughness with height, width, and length normalized to unity, while the ratio of roughness length to width b_1/c_1 appears only in the second boundary problem.

3. For narrow roughness ($b > c$) in region 3 using the equations for continuity and conservation of transverse momentum together with Eq. (1.1) we can obtain values for the vertical and transverse velocities and transverse pressure gradient

$$v \sim O(\varepsilon/b^{1/3}),\ w \sim O(c/b^{2/3}),\ \partial p/\partial z \sim w\,\partial w/\partial z \sim O(c/b^{4/3}). \tag{3.1}$$

If the pressure perturbation is created by interaction of the roughness with a uniform incident flow, then in the perturbed region with characteristic dimensions $\varepsilon < x \sim b \lesssim 1$, $\varepsilon < y \sim z \sim c < b$ flow function perturbations are induced:

$$\Delta u \sim O(a/c),\ v \sim w \sim O(a/b),\ \partial p/\partial z \sim u\,\partial w/\partial x \sim O(a/b^2).$$

Comparing this estimate for $\partial p/\partial z$ with Eq. (3.1) we find that the compensation regime for flow over narrow roughness is realized at

$$a \sim O(\varepsilon b^{1/3}),\ \varepsilon^{3/2} < b < 1,\ a < c < \min(b,\ \varepsilon/b^{1/3}). \tag{3.2}$$

If the pressure perturbation decays in the transverse direction at a distance $z \sim O(c)$, then from Eq. (3.1)

$$\Delta p \sim O(c^2/b^{4/3}), \tag{3.3}$$

i.e., narrow roughnesses induce a pressure perturbation which is $(b/c)^2$ smaller than for broad roughness [see the Δp estimate of Eq. (2.1)].

However, for narrow roughnesses, Eq. (3.2), the maximum roughness dimension is the length b, so that a situation is possible in which the pressure perturbation decays in the transverse direction only at a distance $z \sim O(b)$. Then it becomes necessary also to consider perturbed region 4, which is wider than region 3, with characteristic dimensions $x \sim z \sim O(b)$, $y \sim O(a)$, in which a pressure perturbation

$$\Delta p \sim O(c/b^{1/3}) \tag{3.4}$$

is introduced, i.e., (b/c) times greater than the perturbation of Eq. (3.3).

The estimates of Eqs. (3.1)-(3.4) permit us to introduce in region 3 the following independent variables and asymptotic flow function expansions:

$$\begin{gathered} x = bx_3, \quad y = \varepsilon b^{1/3} y_3, \quad z = cz_3, \\ u = b^{1/3}u_3 + \ldots, \; v = (\varepsilon/b^{1/3})v_3 + \ldots, \; w = (c/b^{2/3})w_3 + \ldots, \\ \Delta p = (c/b^{1/3})p_{31} + (c^2/b^{4/3})p_{32} + \ldots, \quad h = h_w + b^{1/3}h_3 + \ldots, \\ \rho = \rho_w + \ldots, \quad \mu = \mu_w + \ldots \end{gathered} \tag{3.5}$$

Substitution of the expansions of Eq. (3.5) in the Navier–Stokes equations and performance of the limiting transition $\varepsilon \to 0$, $\varepsilon^{3/2} < c < b < 1$ shows that in the first approximation the flow in region 3 is described by the equations of a Prandtl spatial boundary layer for an incompressible gas

$$\begin{gathered} \frac{\partial u_3}{\partial x_3} + \frac{\partial v_3}{\partial y_3} + \frac{\partial w_3}{\partial z_3} = 0, \quad \frac{\partial p_{31}}{\partial y_3} = \frac{\partial p_{32}}{\partial y_3} = \frac{\partial p_{31}}{\partial z_3} = 0, \\ \rho_w\left(u_3\frac{\partial u_3}{\partial x_3} + v_3\frac{\partial u_3}{\partial y_3} + w_3\frac{\partial u_3}{\partial z_3}\right) + \left(\frac{c}{b}\right)\frac{\partial p_{31}}{\partial x_3} + \left(\frac{c}{b}\right)^2\frac{\partial p_{32}}{\partial x_3} = \mu_w\frac{\partial^2 u_3}{\partial y_3^2}, \\ \rho_w\left(u_3\frac{\partial w_3}{\partial x_3} + v_3\frac{\partial w_3}{\partial y_3} + w_3\frac{\partial w_3}{\partial z_3}\right) + \frac{\partial p_{32}}{\partial z_3} = \mu_w\frac{\partial^2 w_3}{\partial y_3^2}, \\ \rho_w\left(u_3\frac{\partial h_3}{\partial x_3} + v_3\frac{\partial h_3}{\partial y_3} + w_3\frac{\partial h_3}{\partial z_3}\right) = \frac{\mu_w}{\Pr}\frac{\partial^2 h_3}{\partial y_3^2}. \end{gathered} \tag{3.6}$$

If the pressure perturbation attenuates in the transverse direction at a distance $z \sim O(c)$, then $p_{31} \equiv 0$, $p_{32} = p_3$, and the solution of system (3.6) must satisfy internal and initial boundary conditions (2.5)-(2.7).

To find the external boundary conditions as $y_3 \to \infty$ it is again necessary to consider perturbed region 2, the characteristic thickness of which $y \sim O(c)$ at $\varepsilon^{3/2} < c < \varepsilon$ or $y \sim O(\varepsilon)$ at $\varepsilon \lesssim c < \min(b, \varepsilon/b^{1/3})$. Introducing corresponding independent variables and asymptotic flow function expansions and performing the necessary merger of the expansions in regions 2 and 3, we have external boundary conditions of the form of Eq. (2.11)

$$\begin{gathered} u_3 \to Ay_3, \quad h_3 \to By_3, \quad A\rho_w v_3 + \left(\frac{c}{b}\right)\frac{\partial p_{31}}{\partial x_3} + \left(\frac{c}{b}\right)^2\frac{\partial p_{32}}{\partial x_3} \to 0, \\ w_3 \to 0 \quad (y_3 \to \infty). \end{gathered} \tag{3.7}$$

Here $p_{31} \equiv 0$, $p_{32} = p_3$, if the pressure perturbation attenuates in the transverse direction at a distance $z \sim O(c)$.

If in the boundary problem of Eqs. (2.5)-(2.7), (3.6), and (3.7), describing the compensation regime of flow over narrow roughnesses, Eq. (3.2), for the condition of pressure perturbation attenuation in the transverse direction at a distance $z \sim O(c)$, we take $b \sim c$, it will be identical to boundary problem (2.4)-(2.7), (2.11). Then using the variables of Eq. (2.12) or (2.19) at

$$p_3 = (\rho_w A^2 a_1^2 c_1^2 / b_1^2)\, p \qquad (3.8)$$

in the first case or at

$$p_3 = (\rho_w A^2 a_1^2 c_1^2 / b_1^2)\, \Pi^{2/3} p \qquad (3.9)$$

in the second, we can reduce the problem to the form of Eqs. (2.13)-(2.17) or (2.15)-(2.17), (2.20), (2.21), respectively, at

$$C = (c_1/b_1)^2,\ D = E = 1\ (c_1 \leqslant b_1). \qquad (3.10)$$

Equations (3.8), (3.9) show that in the case of flow over narrow roughness the pressure perturbation induced is $(b_1/c_1)^2$ times smaller than for flow over nonnarrow roughness [compare Eqs. (2.12) and (2.19)].

As $C \to 0$ in the boundary problem under study terms with $\partial p/\partial x$ vanish and its solution describes flow over narrow roughness ($c_1 \ll b_1$).

It is obvious that Eqs. (2.22)-(2.25) are valid here, and that it is possible to divide the complete boundary problem into two parts, the first of which describes flow over a normalized roughness, while the second defines the pressure perturbation, and contains the square of the roughness length $C = (c_1/b_1)^2$ as a similarity parameter.

4. Since for flow over narrow roughness the pressure perturbation attenuates in the transverse direction at a distance $z \sim O(b)$, boundary condition (2.7) will not be valid for region 3, and it becomes necessary to consider region 4. The missing boundary conditions for region 3 as $z_3 \to \pm\infty$ can then be found from merger conditions for the asymptotic expansions of the flow functions in regions 3 and 4.

However, at $b \gtrsim c$ the boundary problem for flow over roughness can immediately be obtained if in Eqs. (2.4)-(2.7), (2.11) or Eqs. (2.5)-(2.7), (3.6), (3.7), we take $b \sim c$ and introduce new variables (without indices):

$$\begin{gathered} x_3 = b_1 x,\ y_3 = a_1 y,\ z_3 = b_1 z, \\ u_3 = A a_1 u,\quad v_3 = (A a_1^2/b_1)\, v,\quad w_3 = A a_1 w, \\ p_3 = (\rho_w A^2 a_1^2 c_1/b_1)\, p,\quad h_3 = B a_1 h \end{gathered} \qquad (4.1)$$

or

$$\begin{gathered} u_3 = A a_1 \Pi^{1/3} u,\quad v_3 = (A a_1^2/b_1)\, \Pi^{2/3} v,\quad w = A a_1 \Pi^{1/3} w, \\ p_3 = (\rho_w A^2 a_1^2 c_1/b_1)\, \Pi^{2/3} p,\quad h_3 = B a_1 \Pi^{1/3} h, \\ x_3 = b_1 x,\ y_3 = a_1 \Pi^{1/3} y,\ z_3 = b_1 z. \end{gathered} \qquad (4.2)$$

Then the boundary problem takes on the form of Eqs. (2.13)-(2.17) or Eqs. (2.15)-(2.17), (2.20), (2.21), respectively, for

$$C = D = c_1/b_1,\ E = b_1/c_1\ (c_1 \leqslant b_1). \qquad (4.3)$$

It is obvious that in this case the pressure perturbation is only (b_1/c_1) times smaller than for flow over nonnarrow roughness [compare Eqs. (2.12) and (2.19)], while the dimensionless shear stress components τ_{xy}, τ_{yz} and the thermal flux q can be calculated with the expressions

$$\tau_{xy} = \partial u/\partial y,\ \tau_{yz} = \partial w/\partial y,\ q = \partial h/\partial y. \qquad (4.4)$$

For the case of flow over roughness considered here, Eqs. (2.23)-(2.25) are also valid, and again it is possible to separate the complete boundary problem into two. Only now the boundary problem for determination of velocity components and enthalpy contain as a similarity parameter the ratio of the roughness length to its width — $E = b_1/c_1$.

In the limiting case as $C, D \to 0$, $E \to \infty$, $c_1 \ll b_1$ change in roughness form in the transverse direction is described by a delta function, which indicates the necessity of introducing two different scales in the transverse direction: $z \sim O(c)$ and $z \sim O(b)$, $c < b$. Therefore, we will consider again the perturbed region 4 with characteristic dimensions $x \sim z \sim O(b)$,

$y \sim O(\varepsilon b^{1/3})$, in which the pressure perturbation is defined by Eq. (3.4), while the transverse velocity w is given by Eq. (3.1). Then from the continuity equation we obtain estimates for the perturbations in longitudinal and vertical velocities $\Delta u \sim O(c/b^{2/3})$, $v \sim O(\varepsilon c/b^{4/3})$, while in region 4 we introduce the following independent variables and asymptotic expansions of the flow functions:

$$x_4 = x_3 = x/b,\ y_4 = y_3 = y/\varepsilon b^{1/3},\ z_4 = z/b,$$
$$u = b^{1/3}Ay_4 + (c/b^{2/3})u_4 + \ldots,\ v = (\varepsilon c/b^{4/3})v_4 + \ldots, \tag{4.5}$$

$$w = (c/b^{2/3})w_4 + \ldots,\ \Delta p = (c/b^{1/3})p_4 + \ldots,\ h = h_w + b^{1/3}By_4 + (c/b^{2/3})h_4 + \ldots,$$
$$\rho = \rho_w + \ldots,\ \mu = \mu_w + \ldots$$

Substitution of the expansion of Eq. (4.5) in the Navier–Stokes equation and carrying out of the limiting transition as $\varepsilon \to 0$, $\varepsilon^{3/2} < c < b \leq 1$ reveals that in the first approximation the flow in region 4 is described by equations of a Prandtl spatial boundary layer for an incompressible gas linearized relative to the incident flow ($u = b^{1/3}Ay_4$)

$$\frac{\partial u_4}{\partial x_4} + \frac{\partial v_4}{\partial y_4} + \frac{\partial w_4}{\partial z_4} = 0,\quad \rho_w\left(Ay_4\frac{\partial u_4}{\partial x_4} + Av_4\right) + \frac{\partial p_4}{\partial x_4} = \mu_w\frac{\partial^2 u_4}{\partial y_4^2}, \tag{4.6}$$
$$\frac{\partial p_4}{\partial y_4} = 0,\quad \rho_w Ay_4\frac{\partial w_4}{\partial x_4} + \frac{\partial p_4}{\partial z_4} = \mu_w\frac{\partial^2 w_4}{\partial y_4^2},$$
$$\rho_w\left(Ay_4\frac{\partial h_4}{\partial x_4} + Bv_4\right) = \frac{\mu_w}{\Pr}\frac{\partial^2 h_4}{\partial y_4^2}.$$

Since region 4 is wider than the roughness in order of magnitude, the adhesion and nonpenetration conditions must now be satisfied on the surface of a plane plate

$$u_4 = v_4 = w_4 = h_4 = 0\ (y_4 = 0). \tag{4.7}$$

The initial boundary conditions as $x_4 \to -\infty$ or $z_4 \to \pm\infty$ are again obtained by merging the solution in region 4 with the near-the-wall portion of the boundary layer on the plate

$$u_4,\ v_4,\ w_4,\ p_4,\ h_4 \to 0\ (x_4 \to -\infty,\ z_4 \to \pm\infty), \tag{4.8}$$

while the external boundary conditions are found from merger of the solution for region 2 [using expansions of the form of Eq. (2.8) or (2.9) for $c \sim b$]

$$u_4,\ w_4,\ h_4 \to 0,\ A\rho_w v_4 + \partial p_4/\partial x_4 \to 0\ (y_4 \to \infty). \tag{4.9}$$

By merging the expansions of Eqs. (3.5) and (4.5) we obtain insufficient initial boundary conditions for region 3 as $z_3 \to \pm\infty$

$$u_3 \to Ay_3,\ v_3 \to 0,\ \partial w_3/\partial z_3 \to 0,\ h_3 \to By_3\ (z_3 \to \pm\infty), \tag{4.10}$$

i.e., flow over narrow thin roughness, Eq. (3.2), where the pressure perturbation attenuates in the transverse direction at a distance $z \sim O(b)$, is described in region 3 by a solution of the boundary problem of Eqs. (2.5), (2.6) (here by p_3 we must understand p_{32}), (3.6), (3.7), and (4.10), which as $z_3 \to \pm\infty$ defines the transverse pressure gradient distribution

$$\partial p_{32}/\partial z_3 = \pm G(x_3)\,\rho_w A^2 a_1^2 c_1/b_1^2\quad (z_3 \to \pm\infty). \tag{4.11}$$

For region 4, merger of the expansions of Eqs. (3.5) and (4.5) with use of Eq. (4.11) yields

$$\partial p_4/\partial z_4 = \pm G(x_4)\,\rho_w A^2 a_1^2 c_1/b_1^2\quad (z_4 \to \pm 0) \tag{4.12}$$

and the solution of the boundary problem (4.6)-(4.9), (4.12) in region 4 defines

$$p_4 = p_{31}(x_3)\ (z_4 \to 0). \tag{4.13}$$

In the new variables (without indices) of Eqs. (2.12), (3.8) at $p_3 = p_{32}$ or Eqs. (2.19), (3.9) at $p_3 = p_{32}$ the boundary problem of Eqs. (2.5), (2.6) ($p_3 = p_{32}$), Eqs. (3.6), (3.7), and (4.10) for region 3 takes on the form of Eqs. (2.13)-(2.16) and

$$u,\ h \to y,\ v,\ \partial w/\partial z \to 0\ (z \to \pm\infty) \tag{4.14}$$

or Eqs. (2.15), (2.16), (2.20), (2.21), (4.14) at

$$C = 0,\ D = E = 1, \tag{4.15}$$

where Eqs. (2.22)-(2.25) are obviously valid and it is again possible to divide the complete boundary problem into two, the second of which defines the pressure perturbation in region 4 [see Eqs. (4.11), (4.12)].

In region 4 for the boundary problem of Eqs. (4.6), (4.9), (4.12) we again introduce new variables (without indices)

$$\begin{gathered} x_4 = b_1 x,\ y_4 = a_1 \Pi^{1/3} y,\ z_4 = b_1 z, \\ u_4 = (A a_1 c_1/b_1)\, u, \quad v_4 = (A a_1^2 c_1/b_1^2)\, \Pi^{1/3} v, \quad w_4 = (A a_1 c_1/b_1)\, w, \\ p_4 = (\rho_w A^2 a_1^2 c_1/b_1)\, \Pi^{1/3} p, \quad h_4 = (B a_1 c_1/b_1)\, h, \end{gathered} \tag{4.16}$$

and obtain

$$\begin{gathered} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad y\frac{\partial u}{\partial x} + v + C\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0, \\ y\frac{\partial w}{\partial x} + D\frac{\partial p}{\partial z} = \frac{\partial^2 w}{\partial y^2}, \quad y\frac{\partial h}{\partial x} + v = \frac{1}{\Pr}\frac{\partial^2 h}{\partial y^2}, \\ u = v = w = h = 0\ (y = 0), \\ u,\ v,\ w,\ p,\ h \to 0\ (x \to -\infty,\ z \to \pm\infty), \\ u,\ w,\ h \to 0,\ v + C\partial p/\partial x \to 0\ (y \to \infty), \\ \partial p/\partial z = \pm G(x)\Pi^{-1/3}\ (z \to \pm 0) \end{gathered} \tag{4.17}$$

at

$$C = D = 1. \tag{4.18}$$

Here it is also possible to divide the complete boundary problem of Eq. (4.17) into two, the second of which defines the pressure perturbation in region 3 [see Eq. (4.13)].

5. As $\Pi \to \infty$ the boundary problems describing the various cases of the compensation regime of flow over fine roughness on the surface of a plane plate permit linearization with respect to the small parameter $\lambda = \Pi^{-1/3} \ll 1$:

$$\begin{gathered} u = y + \lambda U + \ldots,\ v = \lambda V + \ldots,\ w = \lambda W + \ldots, \\ p = \lambda P + \ldots,\ h = y + \lambda H + \ldots \end{gathered} \tag{5.1}$$

In the new variables of Eq. (5.1) from Eqs. (2.23), (2.24) and boundary conditions (2.15)-(2.17), (2.21) and (2.25) for the dynamic portion of the problem we obtain

$$\begin{gathered} C\frac{\partial^2 P}{\partial x^2} + D\frac{\partial^2 P}{\partial z^2} = \frac{\partial F}{\partial y} \quad (y = 0), \\ y\frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial y^2}, \quad P = P(x, z), \quad F(x, y, z) = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z}\right), \\ P,\ F \to 0\ (x \to -\infty),\ P \to 0\ (x \to \infty), \\ F \to 0, \quad \int_0^y F dy \to \frac{\partial f(x, Ez)}{\partial x} \quad (y \to \infty), \quad P \to 0 \quad (z \to \pm\infty), \end{gathered} \tag{5.2}$$

where again when necessary we use the natural condition of attenuation of the pressure perturbation far from the roughness.

At values of the coefficients C, D, E from Eq. (2.18) boundary problem (5.2) is convenient for study of flow around various nonnarrow roughnesses ($b_1 \leq c_1$), while propagation of the pressure perturbation is determined by an elliptical type equation. In the limiting case of wide roughness ($b_1 \ll c_1$, $D \to 0$, $W \to 0$) the solution of boundary problem (5.2) will describe flow over individual roughness sections at z = const, while the pressure perturbations are found by solution of the equation

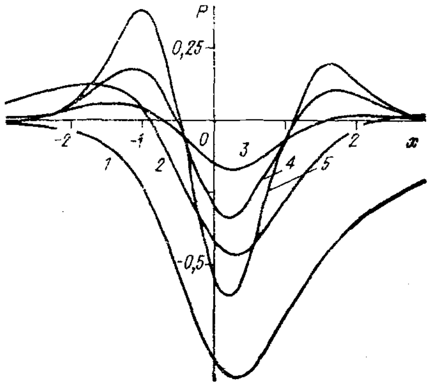

Fig. 1

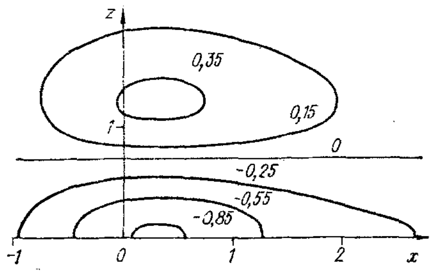

Fig. 2

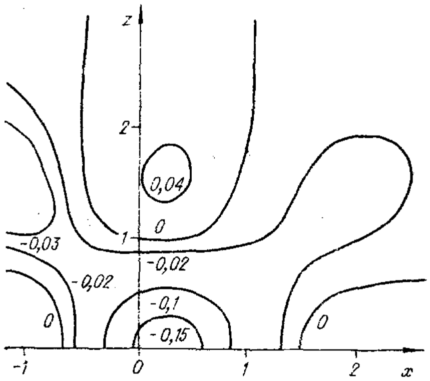

Fig. 3

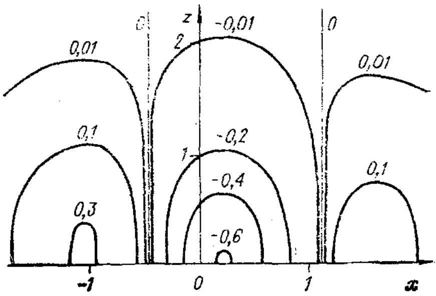

Fig. 4

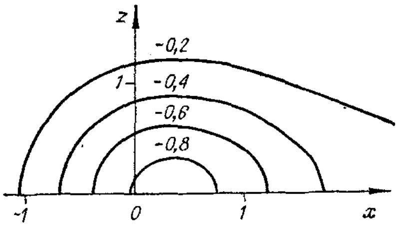

Fig. 5

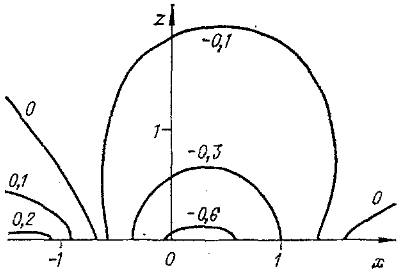

Fig. 6

$$\partial P/\partial x = \partial^2 U/\partial y^2 \ (y = 0),$$

for which transmission of perturbations up the flow does not occur (see also [2]).

For values of the coefficients C, D, E from Eq. (3.10) boundary problem (5.2) permits a solution for narrow roughness ($c_1 < b_1$), where the pressure perturbation attenuates in the transverse direction at a distance equal in order of magnitude to the characteristic width of the roughness. Let the form of the roughness $f(x, z) = f_1(x)f_2(z)$; then $F = f_2(z)F_1(x, y)$, and for $c_1 \ll b_1$ the pressure perturbation is defined by solution of the equation

$$\partial^2 P/\partial z^2 = f_2(z)\partial F_1/\partial y \ (y = 0), \tag{5.3}$$

which will satisfy the attenuation conditions as $z \to \pm\infty$, if

$$\int_{-\infty}^{\infty} f_2(z)\, dz = 0. \tag{5.4}$$

The form of the function $f_2(z)$ defines the change in the pressure perturbation in the transverse direction, while solution of a parabolic-type equation $F_1(x, y)$ defines the longitudinal change, i.e., in this limiting case also perturbations are not transported up the flow.

A numerical solution of boundary problem (5.2) for condition (5.4) was obtained for a roughness $f(x, z) = \exp(-x^2 - z^2)(1 - 2z^2)$. Figure 1 shows pressure perturbation distributions $P(x, 0)$ along the line of symmetry $z = 0$. For $c_1 \gg b_1$ each section of the roughness $z = \text{const}$ induces a pressure perturbation independent of the other sections, and a protuberance of the form $\exp(-x^2)$ in the infrasonic wall portion of the boundary layer on the plate produces rarefaction (curve 1). At $c_1 = 4b_1$, for example, the roughness produces smaller pressure perturbations than at $c_1 \gg b_1$, and because of perturbation propagation up the flow positive pressure perturbations appear at $x < 0$ (curve 2). As the ratio c_1/b_1 decreases, the intensity of the pressure perturbation falls [at $c_1/b_1 < 1$ the values of the pressure perturbation are multiplied by $(b_1/c_1)^2$], and pressure perturbation propagation maintains an elliptical character (curves 3 and 4 for $c_1/b_1 = 1$ and 0.25, respectively). In the limiting case $b_1 \gg c_1$ the distribution is described by the expression (line 5) $P(x, z) = -0.5\exp(-z^2)\partial F_1/\partial y$ $(y = 0)$.

The isobars shown in Figs. 2-4 ($b_1 \ll c_1$, $b_1 = c_1$, and $b_1 \gg c_1$, respectively) reveal the complex character of pressure perturbation propagation at $b_1 = c_1$ and the degeneration of the flow in the longitudinal or transverse directions for $b_1 \ll c_1$ or $b_1 \gg c_1$.

If condition (5.4) is not satisfied, then for narrow roughness ($c_1 < b_1$) the pressure perturbation attenuates in the transverse direction at a distance comparable to the characteristic length of the roughnesses, and the solution of boundary problem (5.2) with coefficient values C, D, E from Eq. (4.3) is valid.

In the limiting case ($c_1 \ll b_1$) in region 3 the pressure perturbation distribution is described by the solution of boundary problem (5.2) with C, D, E from Eq. (4.15). Obviously it is unnecessary here to satisfy the pressure perturbation attenuation conditions at $z \to \pm\infty$, and there is no transport of perturbations up the flow. Then for roughnesses with longitudinal symmetry we obtain from Eq. (5.2)

$$G(x) = \pm \frac{\partial F_1}{\partial y} \int_0^\infty f_2(z)\, dz \quad (y = 0). \tag{5.5}$$

Pressure perturbations attenuate in region 4, for which it follows from Eqs. (4.17), (5.5) that

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial z^2 = 0,$$
$$p \to 0 \ (x \to \pm\infty,\ z \to \pm\infty),\ \partial p/\partial z = \pm G(x) \ (z \to \pm 0). \tag{5.6}$$

We will note that in the case of narrow roughness studied here ($b_1 \gg c_1$) the pressure perturbations induced are (b_1/c_1) times larger than when condition (5.4) is satisfied.

Numerical solutions were obtained here for a roughness $f(x, z) = \exp(-x^2 - z^2)$. The pressure perturbation distributions $P(x, 0)$ repeat in principle the curves presented in Fig. 1, i.e., transport of pressure perturbations up the flow occurs here also.

Figures 5 and 6 show isobars for roughnesses of the limiting types $b_1 \ll c_1$ and $b_1 \gg c_1$, respectively. In the first case a convex roughness in the infrasonic flow produces rarefaction over the entire flow field. In the case of narrow roughness pressure perturbation propagation is of a complex elliptical character, analogous to that shown in Fig. 6.

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DISTRIBUTION OF TURBULENCE CHARACTERISTICS IN A CHANNEL WITH INTENSIVE INJECTION

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UDC 532.517.4

A large number of studies (see, e.g., [1]) have been devoted to aspects of the distribution of flow characteristics in channels with injection. A theoretical analysis of the corresponding solution of the Navier–Stokes equations for laminar flow was first made in [2]. Subsequent experimental studies [3-7] showed that with a turbulent flow regime, the profiles of the longitudinal and transverse components of the velocity vector are described well by limit relations (infinitely large Reynolds number for injection) in [2]. This result, evidence of the high degree of stability of the flow, can be attributed to laminarization of the flow as it is accelerated due to distribution of the injection in the channel [8]. Use of the Prandtl model to describe the distribution of the turbulence characteristics in a channel with injection [9] leads to relations which are inconsistent with this fact.

Here we attempt to construct an approximate semiempirical theory to describe flow characteristics based on the $(k - \varepsilon)$-model of turbulence. By numerically integrating the hydrodynamic equations with the $(k - \varepsilon)$-model, we calculated flow parameters in a broad range of injection Reynolds numbers. The results of the calculations agree well with the experimental data.

1. We are examining a steady flow of a viscous incompressible fluid in a plane channel (Fig. 1) at a sufficiently large distance from the impermeable left wall. Fluid of the density ρ^0 is injected through the permeable top wall of the channel at a constant velocity $q_b{}^0$. The equations describing the flow and the boundary conditions appear as follows in dimensionless form

$$w\frac{\partial w}{\partial z} + v\frac{\partial w}{\partial y} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(\frac{1}{\mathrm{Re}}\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mathrm{Re}}\frac{\partial w}{\partial y}\right), \tag{1.1}$$
$$w\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\frac{1}{\mathrm{Re}}\frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\mathrm{Re}}\frac{\partial v}{\partial y}\right), \quad \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0,$$

where w and v are averaged values of the components of the velocity vector $\mathbf{q}$ along the axes z and y (see Fig. 1):

$$y = 0\colon\ v = 0 = \partial w/\partial y;\ y = 1\colon\ v = -1,\ w = 0;\ z = 0\colon\ w = v = 0. \tag{1.2}$$

No conditions are imposed on the right boundary because we are studying a self-similar solution of system (1.1). We use the following as the scales of length, velocity, and pressure in (1.1) and (1.2): h^0 is half the width of the channel; $q_b{}^0$ and $\rho^0 q_b{}^{02}$, $\mathrm{Re} = \rho^0 q_b{}^0 h^0/\mu^0$ is the characteristic injection Reynolds number for the problem; μ^0 is the viscosity of the fluid ($\mu^0 = \mu_\ell{}^0 + \mu_t{}^0$, $\mu_\ell{}^0$ and $\mu_t{}^0$ are the laminar and turbulent components).

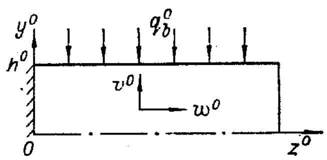

Fig. 1

Biisk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 79-84, September-October, 1987. Original article submitted June 19, 1986.

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